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ELLIPTIC PERIODIC ORBITS NEAR A HOMOCLINIC TANGENCY IN FOUR-DIMENSIONAL SYMPLECTIC MAPS AND HAMILTONIAN SYSTEMS WITH THREE DEGREES OF FREEDOM

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We study bifurcations leading to the appearance of elliptic orbits in the case of four-dimensional symplectic diffeomorphisms (and Hamiltonian flows with three degrees of freedom) with a homoclinic tangency to a saddle-focus periodic orbit.

To Prof. Jürgen Mozer, on his 70th anniversary

1. Introduction and main results.

Two-dimensional area-preserving analytical diffeomorphisms with a structurally unstable heteroclinic cycle were considered in [1]. Such diffeomorphisms are divided into three classes depending on the structure of the set of orbits lying entirely in a small neighborhood of the heteroclinic cycle. It was shown in [1] that diffeomorphisms having infinitely many elliptic islands (in this small neighborhood) are dense among the diffeomorphisms of the third class.

In the present paper we consider the problem on the existence of elliptic periodic orbits for *four-dimensional symplectic diffeomorphisms* (and for three degrees of freedom Hamiltonian flows) with homoclinic tangencies.

The structurally stable periodic orbits of four-dimensional symplectic diffeomorphisms, or of three degrees of freedom Hamiltonian flows in a fixed energy level, can be of the three following types [3]:

1) *saddle*, i.e. such that have one pair of multipliers inside the unit circle and the other pair outside; among saddle periodic orbits one may distinguish *the saddles* (the periodic orbits with real multipliers $\nu_1 = \gamma_1$, $\nu_2 = \gamma_2$, $\nu_3 = \gamma_1^{-1}$, $\nu_4 = \gamma_2^{-1}$ where $|\gamma_j| < 1$) and *the saddle-foci* (the periodic orbits with

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complex multipliers $\nu_{1,2} = \lambda e^{\pm i\varphi}$, $\nu_{3,4} = \lambda^{-1} e^{\pm i\varphi}$ where $0 < \lambda < 1$ and $0 < \varphi < \pi$);

2) *1-elliptic*, i.e. such that have a pair of real multipliers and a pair of multipliers on the unit circle:

$\nu_1 = \gamma$, $\nu_2 = \gamma^{-1}$, $\nu_{3,4} = e^{\pm i\omega}$ where $|\gamma| < 1$ and $0 < \omega < \pi$;

3) *2-elliptic* or simply *elliptic*, i.e. such that have all multipliers on the unit circle: $\nu_{1,2} = e^{\pm i\omega_1}$, $\nu_{3,4} = e^{\pm i\omega_2}$ where $0 < \omega_{1,2} < \pi$ and $\omega_1 \neq \omega_2$.

In the symplectic polar coordinates, the map near a 2-elliptic fixed point¹ can be written in the following (Birkhoff) normal form if there is no strong resonances (i.e. $\omega_1 \neq \omega_2$, $\omega_1 \neq 2\omega_2$, $\omega_1 \neq 3\omega_2$, $\omega_2 \neq 2\omega_1$, $\omega_2 \neq 3\omega_1$, $\omega_1 + \omega_2 \neq \pi$, $\omega_1 + 2\omega_2 \neq 2\pi$, $2\omega_1 + \omega_2 \neq 2\pi$, $3\omega_1 \pm \omega_2 \neq 2\pi$, $3\omega_2 \pm \omega_1 \neq 2\pi$, $\omega_{1,2} \neq 2\pi/3, \pi/2$):

$$\bar{\rho} = \rho + o(\rho^2), \quad \bar{\theta} = \theta + \omega + \Omega\rho + o(\rho)$$

where $\rho \in R^2$, $\theta \in T^2$, $\omega = (\omega_1, \omega_2)$ and Ω is a (2×2) -matrix. If Ω is non-degenerate (i.e. $\det \Omega \neq 0$), then the Kolmogorov-Arnold-Moser theory is applied to this map at small ρ , which guarantees the existence of a large (in measure) set of invariant tori near $\rho = 0$ (the map must be at least C^5 smooth [4]).

Such elliptic point is called *generic*. Unlike the two-dimensional case, a generic 2-elliptic point may be unstable in the usual (Lyapunov) sense. However, for the majority of initial conditions near such point (the initial conditions on the invariant tori) the orbit never escapes the small neighborhood of the fixed point. Thus, we have here some form of stability.

The question now is how elliptic periodic orbits can appear in chaos for four-dimensional maps. Note that there cannot be a straightforward analogy with the two-dimensional results. Indeed, *unfolding a homoclinic tangency to a saddle (i.e. when multipliers are real) can not produce 2-elliptic periodic orbits if the transversality conditions of [5, 15, 6] are satisfied.*

These conditions guarantee the existence of a uniform partially hyperbolic structure in a small neighborhood of the homoclinic tangency, which essentially means the existence of a positive Lyapunov exponent for every orbit and therefore prevents of appearance of stable periodic orbits.

Another possibility which we explore in this paper is the homoclinic tangency to a saddle-focus. We will consider maps and flows simultaneously, so "a system X " will refer to a four-dimensional symplectic map or a three-degrees of freedom Hamiltonian flow on a fixed level of the Hamiltonian function H . We will also consider finite-parameter families of such systems. The natural one-parameter family of the flows under consideration is the family of flows on different levels of the same Hamiltonian. However, other choices are also allowed (say, we may fix the energy level but consider the Hamiltonians which depend on parameters, etc.).

Let a C^r -smooth ($r \geq 2$) system X_0 satisfy the following conditions.

A. X_0 has a periodic orbit L_0 which is a saddle-focus with the multipliers $\nu_{1,2} = \lambda_0 e^{\pm i\varphi_0}$, $\nu_{3,4} = \lambda_0^{-1} e^{\pm i\varphi_0}$, where $0 < \lambda_0 < 1$.

Obviously, any close system will have a saddle-focus periodic orbit L close to L_0 .

B. The stable and unstable invariant manifolds W^s and W^u of L_0 have a *simplest* tangency at the points of some homoclinic orbit Γ .

The condition of the "simplicity" of the tangency (or the quasi-transversal intersection) reads as follows. Let $E_M W$ denote the tangent space to a manifold W at some point $M \in W$. Let M^* be any point on the homoclinic orbit Γ . Then we require

B.1. $\dim(E_{M^*} W^s \cap E_{M^*} W^u) = 1$

for the maps, or

B.1'. $\dim(E_{M^*} W^s \cap E_{M^*} W^u) = 2$

for the flows²; and

B.2. The tangency of W^s and W^u is quadratic at M^* .

¹The study of a periodic orbit can always be reduced to the study of a fixed point: by taking some power of the map or by considering the Poincaré map on a cross-section to the flow.

²The dimension increases here because the phase velocity vector of the flow belongs to the tangent of any invariant manifold.

In turn, one can give a coordinate formulation of conditions B.1–B.2 . Namely, the quasi-transversal intersection means that in some local C^2 -coordinates (x_1, x_2, y_1, y_2) near the intersection point M^* (for the flows these coordinates are on a cross-section $S_* \ni M^*$) one has that

$$W^s = \{y_1 = 0, y_2 = 0\} \quad \text{and} \quad W^u = \{x_2 = 0, y_1 = x_1^2\} \quad (1)$$

(for the flows (1) takes place for $W^s \cap S_*$ and $W^u \cap S_*$, respectively).

We will consider C^r -smooth two-parameter families $X_{\mu\varphi}$. Here, the dependence on μ is assumed to be generic in the sense that the orbit Γ of homoclinic tangency bifurcates into two transverse homoclinic orbits at, say, $\mu < 0$ whereas at $\mu > 0$ there is no homoclinic intersections close to Γ ; moreover, the distance between W^u and W^s near any point of Γ grows with non-zero velocity as μ grows towards positive values (see (50)). In other words, μ can be viewed as a standard splitting parameter. The parameter φ is the argument of the multipliers of the saddle-focus periodic orbit L , varying near $\varphi = \varphi_0$.

Let U be some sufficiently small neighborhood of $L_0 \cup \Gamma$. It is the union of a small neighborhood U_0 of L_0 and a neighborhood U_1 of $\Gamma \setminus U_0$. When we consider the flows, U_0 is a solid torus and U_1 is a handle glued to U_0 . In the case of maps, U_0 is the union of a finite number m of disjoint balls centered at the points of L_0 (where m is the period of $L_0 = \{O_1, \dots, O_m\}$, $O_{i+1} = X_0 O_i$, $O_1 = X_0 O_m$) and U_1 is the union of a finite number of small neighborhoods of those points of Γ which do not belong to U_0 .

The main problem we study for the family $X_{\mu\varphi}$ concerns the bifurcations of *single circuit* periodic orbits in U . The term "single circuit" refers to those periodic orbits which visit each connected component of $U_1 = U \setminus U_0$ only once in the case of maps³ or those which have only one connected component in the intersection with U_1 in the case of flows.

The main result is given by the following theorem.

Theorem 1. *On the parameter plane (μ, φ) there exists an infinite sequence of regions Δ_i accumulating at the point $(\mu, \varphi) = (0, \varphi_0)$, such that the system $X_{\mu\varphi}$ has a 2-elliptic (generic if $r \geq 5$) single circuit periodic orbit at $(\mu, \varphi) \in \Delta_i$.*

The proof of the theorem is based on the reduction of the first return map in a small neighborhood of a single circuit periodic orbit to some universal form, by means of a linear rescaling of coordinates and parameters. This gives a kind of "asymptotic normal form" for the bifurcations of single circuit orbits, in the same way as it was done in [5, 15] for general dynamical systems with homoclinic tangencies. For the systems under consideration, such a normal form is a four-dimensional analogue of Hénon map⁴

$$\begin{aligned} \bar{x}_2 &= x_1, & \bar{x}_1 &= y_2, & \bar{y}_2 &= y_1, \\ \bar{y}_1 &= -x_2 + M_1(x_1 + y_1) + M_2 - y_2^2. \end{aligned} \quad (2)$$

Here M_1 and M_2 are, essentially, rescaled parameters μ and $(\varphi - \varphi_0)$ (see (74)). They are no longer small and can take arbitrarily finite values. We first study bifurcations in this map and find the regions in (M_1, M_2) -plane which correspond to 2-elliptic periodic orbits. Then, returning to the non-rescaled parameters, we determine the location of the corresponding regions in (μ, φ) -plane.

The significance of the reduction to the asymptotic normal form is, as it follows from Lemma 4 in Section 5, that *any generic dynamics, or any generic codimension-one and codimension-two bifurcations in the map (2) appear in the unfolding of the homoclinic tangency to a saddle-focus in the four-dimensional symplectic maps*. A list of these bifurcations is given in Section 6.

The paper is organized as follows. In Section 2 some necessary information about symplectic maps is collected. In Section 3 we consider a four-dimensional symplectic map T_0 in a small neighborhood

³It is clear that each point of a single circuit periodic orbit is a fixed point for the first return map into the corresponding connecting component.

⁴It preserves the symplectic form $x_1 \wedge (y_1 - M_1 y_2) + x_2 \wedge y_2$.

of the saddle-focus periodic orbit. We show that such maps can be reduced locally to some special form by a canonical coordinate transformation of class C^{r-1} (Lemma 1) and that all iterations of T_0 , when brought to this form, are uniformly close to the iterations of the linearized map (Lemmas 2 and 3). In Section 4 the global map T_1 is considered (i. e. the map by the orbits of X_μ near the piece of the homoclinic orbit $\Gamma \cap U_1$): by a symplectic rotation of coordinates which preserves the special form of the local map T_0 we bring T_1 to some standard form. In Section 5 the first return maps $T_i \equiv T_1 T_0^i$ are studied: an affine transformation of coordinates and parameters makes the first return map asymptotically (as $i \rightarrow +\infty$) close to the map (2). In Section 6 the bifurcations of the fixed points of this Hénon-like map are studied and the main theorem is finally proved.

2. Basic information on symplectic maps

The following definitions are taken mainly from [3].

Consider an even-dimensional linear space $\mathbb{R}^{2n}$. The *symplectic structure* in $\mathbb{R}^{2n}$ is a nondegenerate antisymmetric bilinear 2-form: $[\xi, \eta] = -[\eta, \xi]$ where $\xi, \eta \in \mathbb{R}^{2n}$. The space $\mathbb{R}^{2n}$ endowed by the symplectic structure $[\cdot, \cdot]$ is called *symplectic linear space*. A linear transformation $\mathcal{L} : \mathbb{R}^{2n} \mapsto \mathbb{R}^{2n}$ is *symplectic*, if it preserves the form $[\cdot, \cdot]$, i. e. $[\mathcal{L}\xi, \mathcal{L}\eta] = [\xi, \eta]$.

Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ be the symplectic basis, i. e. an orthonormal basis which satisfies

$$\begin{aligned} [\alpha_i, \beta_i] &= -[\beta_i, \alpha_i] = 1, \\ [\alpha_i, \alpha_j] &= 0 \text{ and } [\beta_i, \beta_j] = 0 \text{ for any } i \text{ and } j, \\ [\alpha_i, \beta_j] &= 0 \text{ at } i \neq j. \end{aligned}$$

The corresponding coordinate frame is called *symplectic*. In symplectic coordinates the form $[\xi, \eta]$ takes the standard form

$$[\xi, \eta] = x \bigwedge y \equiv x_1 \bigwedge y_1 + \dots + x_n \bigwedge y_n$$

where $\xi = (x_1, \dots, x_n, y_1, \dots, y_n)$, $\eta = (x'_1, \dots, x'_n, y'_1, \dots, y'_n)$, and $x_j \bigwedge y_j = x_j y'_j - x'_j y_j$.

In the symplectic basis, a linear symplectic map $\mathcal{L} : (x, y) \mapsto (\bar{x}, \bar{y})$ is defined by a *symplectic matrix* $A : (\bar{x}, \bar{y}) = A(x, y)$, i. e. by a matrix which satisfies

$$A^\top J A = J \quad (3)$$

where $A^\top$ is the transpose to A ,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

and I is the $(n \times n)$ identity matrix.

Recall some properties of symplectic matrices.

Property 1 [3]

The coefficients of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ of a symplectic map are symmetric: if

$$p(\lambda) = a_0 \lambda^{2n} + a_1 \lambda^{2n-1} + \dots + a_{2n},$$

then

$$a_0 = a_{2n}, \dots, a_s = a_{2n-s}, \dots$$

Thus, if ν is an eigenvalue of a symplectic matrix, then ν^{-1} is also an eigenvalue. In particular $\det A = 1$.

Obviously, this property holds for any linear symplectic map, independently of the basis in which it is written down.

Property 2

If A is a symplectic matrix, then $A^\top$ and A^{-1} are also symplectic.

This is an easy consequence of the characteristic Property (3). Indeed, since $J^2 = -I_{2n}$ (where I_{2n} is the $(2n \times 2n)$ identity matrix), we have

$$\begin{aligned} A^\top J A = J &\Rightarrow A^\top J = J A^{-1} \Rightarrow J A^\top J = -A^{-1} \Rightarrow \\ &\Rightarrow A J A^\top J = -I_{2n} \Rightarrow A J A^\top = J \end{aligned}$$

and the symplecticity of $A^\top$ follows. Analogously, for the matrix A^{-1} we have

$$A^\top J A = J \Rightarrow J A = (A^{-1})^\top J \Rightarrow J = (A^{-1})^\top J A^{-1}.$$

Let us now write a symplectic matrix A as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (4)$$

where a , b , c and d are $(n \times n)$ -matrices.

Property 3

1. The inverse to A is given by the following formula:

$$A^{-1} = \begin{pmatrix} d^\top & -b^\top \\ -c^\top & a^\top \end{pmatrix}. \quad (5)$$

2. The matrices a , b , c and d satisfy the following relations:

$$\begin{aligned} 1) & a^\top c = c^\top a, \\ 2) & b^\top d = d^\top b, \\ 3) & d^\top a - b^\top c = I \end{aligned} \quad (6)$$

and

$$\begin{aligned} 1) & ab^\top = ba^\top, \\ 2) & cd^\top = dc^\top, \\ 3) & da^\top - cb^\top = I. \end{aligned} \quad (7)$$

3. For any symmetric $(n \times n)$ -matrix u (i. e. $u^\top = u$), the matrices

$$(c + du)(a + bu)^{-1}, \quad (au + b)(cu + d)^{-1}, \quad (b + ud)^{-1}(a + uc), \quad (ub + d)^{-1}(ua + c)$$

and the inverse to them are symmetric.

Formula (6) follows directly from (3) when the matrix A is written in the block form (4); formula (7) is simply the relation (6) for the matrix $A^\top$. Formula (5) is verified by direct multiplication, with taking identities (6) into account. To prove Property 3.3, note that since u is symmetric, i. e. $u^\top = u$, the matrices $\begin{pmatrix} I & 0 \\ u & I \end{pmatrix}$ and $\begin{pmatrix} I & u \\ 0 & I \end{pmatrix}$ satisfy (6) and, hence, they are symplectic. The product of symplectic matrices is symplectic by definition, therefore the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ u & I \end{pmatrix} = \begin{pmatrix} a + bu & b \\ c + du & d \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} I & u \\ 0 & I \end{pmatrix} = \begin{pmatrix} a & au + b \\ c & cu + d \end{pmatrix}$$

are symplectic too. Thus, they must satisfy the identities (6) which read here, respectively, as $(a+bu)^\top \cdot (c+du) = (c+du)^\top \cdot (a+bu)$ and $(cu+d)^\top \cdot (au+b) = (au+b)^\top \cdot (cu+d)$, but this just means that the matrices, respectively, $(a+bu)(c+du)^{-1}$ (and $(c+du)(a+bu)^{-1}$ also) and $(au+b)(cu+d)^{-1}$ are symmetric indeed. Analogously, multiplying the matrices $\begin{pmatrix} I & 0 \\ u & I \end{pmatrix}$ and $\begin{pmatrix} I & u \\ 0 & I \end{pmatrix}$ on $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from the left, one gets that $(b+ud)^{-1}(a+uc)$ and $(ub+d)^{-1}(ua+c)$ are symmetric as well.

Given a symplectic structure, a diffeomorphism in $\mathbb{R}^{2n}$ is called symplectic if its derivative (at any point) preserves the symplectic form. More generally, symplectic maps on even-dimensional manifolds may be considered where the symplectic form is a differential bilinear antisymmetric form defined on the tangent space. The coefficients of the form may now depend on the point on the manifold. However, the study of the systems we consider in this paper is reduced to the study of symplectic maps in a small neighborhood of a single point. Thus, by virtue of Darboux theorem, we can assume from very beginning that we deal with the standard symplectic structure.

In this case, the Jacobi matrix of the symplectic map satisfies (3) at any point. By definition, symplectic diffeomorphisms form a group (i. e. the product of symplectic maps is symplectic and the inverse to a symplectic map is symplectic either). Thus, dealing with symplectic maps we may freely make symplectic coordinate transformations without losing the structure. Of course, after an arbitrary coordinate transformation, a symplectic diffeomorphism remains symplectic but the symplectic form may change. Therefore, when making non-symplectic coordinate transformations we will always trace out the changes in the symplectic form.

2.1. Symplectic map near a saddle periodic orbit.

Straightening of invariant manifolds

Let a C^r -smooth symplectic diffeomorphism X have a periodic orbit. Let O be a point of this orbit, so $X^m O = O$ for some $m \geq 1$. One may always assume that O is in the origin. Let U be a sufficiently small neighborhood of O . Consider a *local map* $T_0 \equiv X^m|_U$.

We will also consider C^r -smooth flows on a fixed energy level of a Hamiltonian system. If L is a periodic orbit of such system, then the Poincaré map of a small cross-section to L is symplectic. Let O be the point of intersection of L with the cross-section and U be a small neighborhood of O on the cross-section. Then the local map T_0 is just the Poincaré map defined in U .

Let the periodic orbit under consideration be saddle, i. e. we assume that its multipliers (the eigenvalues of the derivative of T_0 at O) do not lie on the unit circle. By Property 1, exactly n multipliers lie inside the unit circle and n lie outside. One may choose a symplectic basis in such a way that in the corresponding symplectic coordinates (x, y) , where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, the spaces $y = 0$ and $x = 0$ will be the eigenspaces of the derivative of T_0 at O , corresponding to the multipliers inside and outside the unit circle, respectively.

In this basis the map T_0 is written as⁵

$$\begin{aligned} \bar{x} &= Bx + F(x, y), \\ \bar{y} &= B^{-\top}y + G(x, y) \end{aligned} \quad (8)$$

where the spectrum of B lies strictly inside the unit circle, and the functions F and G vanish at the origin along with their first derivatives.

The fixed point in the origin has local stable and unstable manifolds W_{loc}^s and W_{loc}^u which are written as $y = h_s(x)$ and $x = h_u(y)$ respectively, where h_s and h_u are C^r -smooth functions such that

$$h_s(0) = 0, \quad \frac{\partial h_s}{\partial x}(0) = 0, \quad h_u(0) = 0, \quad \frac{\partial h_u}{\partial y}(0) = 0.$$

⁵We use the notation $B^{-\top}$ for the inverse to the matrix B transposed.

It is well-known that W_{loc}^s and W_{loc}^u are Lagrange manifolds; when we deal with the standard symplectic form this just means that h_s and h_u are gradients of some smooth functions in $\mathbb{R}^n$. Equivalently, the derivatives of h_s and h_u are symmetric matrices:

$$\frac{\partial h_s}{\partial x} = \left(\frac{\partial h_s}{\partial x} \right)^\top \quad (9)$$

and

$$\frac{\partial h_u}{\partial y} = \left(\frac{\partial h_u}{\partial y} \right)^\top. \quad (10)$$

Recall the proof of (10). The invariance of W_{loc}^u means that if $x = h_u(y)$, then $\bar{x} = h_u(\bar{y})$, i. e.

$$h_u(\bar{y}) = B h_u(y) + F(h_u(y), y) \quad (11)$$

where

$$\bar{y} = B^{-\top} y + G(h_u(y), y).$$

The functional equation (11) is solved by the successive approximation method, i. e. h_u is found as a limit (in C^r -topology) of the sequence of functions $h_n(x)$ defined inductively as

$$h_{n+1}(\bar{y}) = B h_n(y) + F(h_n(y), y), \quad h_0(y) \equiv 0,$$

where

$$\bar{y} = B^{-\top} y + G(h_n(y), y).$$

Correspondingly, the derivative $h'_u(y)$ is the limit of the sequence $u_n(y)$:

$$u_{n+1}(\bar{y}) = \left(\frac{\partial \bar{x}}{\partial x} u_n(y) + \frac{\partial \bar{x}}{\partial y} \right) \left(\frac{\partial \bar{y}}{\partial x} u_n(y) + \frac{\partial \bar{y}}{\partial y} \right)^{-1}, \quad u_0 \equiv 0.$$

The matrix u_0 is clearly symmetric. Therefore, in order to prove (10), it is sufficient to check that once u_n is a symmetric matrix, u_{n+1} is symmetric too, but this follows from the symplecticity of the map $(x, y) \rightarrow (\bar{x}, \bar{y})$, see Property 3.3.

In fact, one may further assume that

$$h_s \equiv 0 \quad \text{and} \quad h_u \equiv 0,$$

i. e. that the local stable and unstable manifolds are straightened. Indeed, condition (10) guarantees that the following coordinate transformation

$$\xi = x - h_u(y), \quad \eta = y$$

is symplectic (because it follows from (10) that the derivative $\begin{pmatrix} I & -h'_u(y) \\ 0 & I \end{pmatrix}$ satisfies (6)). Obviously, it is a C^r -transformation and the local unstable manifold is now given by equation $\xi = 0$. Analogously, a symplectic coordinate transformation straightens the local stable manifold.

After straightening the invariant manifolds, the local map T_0 is written in the following form

$$\begin{aligned} \bar{x} &= Bx + f(x, y)x, \\ \bar{y} &= B^{-\top}y + g(x, y)y \end{aligned} \quad (12)$$

where f and g are C^{r-1} -functions such that

$$f(0, 0) = 0, \quad g(0, 0) = 0.$$

By (6), the symplecticity of T_0 is equivalent to the following relation between f and g :

$$B^{-1}f + g^{\top}B + g^{\top}f = -(g_y y)^{\top}(B + f) - (B^{-1} + g^{\top})f_x x + (f_y x)^{\top}g_x y - (g_y y)^{\top}f_x x, \quad (13)$$

plus the matrices

$$(B^{-1} + (g_y y)^{\top} + g^{\top})f_y x \quad \text{and} \quad (B^{\top} + f^{\top}f + (f_x x)^{\top})g_x y \quad (14)$$

must be symmetric.

3. Four-dimensional symplectic map near a saddle-focus fixed point

Now let the saddle periodic orbit of X be a saddle-focus, i. e. we assume that the multipliers are $\lambda e^{\pm i\varphi}$ and $\lambda^{-1}e^{\pm i\varphi}$ where $0 < \lambda < 1$, $0 < \varphi < \pi$. In this case the matrix B in formula (12) for the local map T_0 is

$$B = \lambda \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}; \quad (15)$$

the variables x and y are now two-dimensional: $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Lemma 1. *There exists a local symplectic C^{r-1} -transformation of coordinates (preserving the form (12) of the map T_0) after which the functions f and g satisfy the following identities:*

$$\begin{aligned} f(0, y) &\equiv 0, & g(0, y) &\equiv 0, \\ g(x, 0) &\equiv 0, & f(x, 0) &\equiv 0. \end{aligned} \quad (16)$$

REMARK 1.

1) The lemma is valid at $r \geq 2$. When $r \geq 3$ the lemma allows one to represent the map T_0 as

$$\bar{x} = Bx + O(\|x\|^2\|y\|), \quad \bar{y} = B^{-\top}y + O(\|y\|^2\|x\|). \quad (17)$$

2) By “the standard method” (using the generating functions) one may prove the existence of C^{r-2} -coordinates desired. We use here a “direct” method: the change is represented in the explicit form and then its symplecticity is proven.

Proof.

Let us make the local transformation

$$x_{new} = (I + \psi(y))x, \quad y_{new} = \Psi(y) \quad (18)$$

where $\psi(0) = 0$, $\Psi(0) = 0$. By Property 3.1, this transformation is symplectic if and only if

$$\Psi'(y) = (I + \psi^{\top}(y))^{-1} \quad (19)$$

and the matrix

$$-(I + \psi(y))^{-1}\psi'(y)x \quad (20)$$

is symmetric for any x and y . The function Ψ is found from (19) as

$$\Psi(y) = \int_0^1 (I + \psi^{\top}(sy))^{-1}y \, ds.$$

One can check that if the matrix (20) is symmetric for any vector x , then such defined Ψ is a smooth function indeed (i. e. $\frac{\partial^2 \Psi}{\partial y_1 \partial y_2} = \frac{\partial^2 \Psi}{\partial y_2 \partial y_1}$).

Let us choose the function ψ^6 such that

$$\psi(\bar{y}) = B\psi(y)[B + f(0, y)]^{-1} - f(0, y)[B + f(0, y)]^{-1} \quad (21)$$

where we denote

$$\bar{y} = [B^{-\top} + g(0, y)]y. \quad (22)$$

To show the existence of a C^{r-1} -smooth function ψ which yields a solution of the functional equation (21), note that ψ satisfies this equation if and only if $z = \psi(y)$ is an invariant manifold of the following map:

$$\begin{aligned} \bar{z} &= Bz[B + f(0, y)]^{-1} - f(0, y)[B + f(0, y)]^{-1}, \\ \bar{y} &= [B^{-\top} + g(0, y)]y \end{aligned} \quad (23)$$

where $z \in \mathbb{R}^{2 \times 2}$. The point $(y = 0, z = 0)$ is the fixed point for this map. The linearized map at this point is

$$\begin{aligned} z &\mapsto BzB^{-1} - f'_y(0, 0)yB^{-1}, \\ y &\mapsto B^{-\top}y. \end{aligned} \quad (24)$$

The linearized map is in the block-triangular form. The eigenvalues which correspond to variables y are the eigenvalues of $B^{-\top}$ and they lie outside the unit circle. The other eigenvalues of (24) lie on the unit circle (because, for any square matrices C_1 and C_2 , the eigenvalues of the operator $z \mapsto C_1 z C_2$ are the pairwise products of the eigenvalues of C_1 and C_2 ; see, for example, [8]; in our case this rule gives $\nu_{1,2} = 1$, $\nu_{3,4} = e^{\pm 2i\varphi}$, see (15)). Such separation of the spectrum of the multipliers of the zero fixed point of the map (23) implies (see more details in [9]) that this map has a unique smooth invariant *strong unstable* manifold of the kind $z = \psi(y)$ where $\psi(0) = 0$. Thus, the function ψ satisfying (21) exists indeed. The smoothness of the strong unstable manifold equals to the smoothness of the corresponding map; since the right-hand side of (23) involves a C^{r-1} -function f , it follows that $\psi \in C^{r-1}$.

To ensure that such taken function ψ defines a smooth symplectic transformation of coordinates by formulas (18), (19), we must verify that the matrix (20) is symmetric. To this aim, note that like an unstable invariant manifold (see Section 2.1), the strong unstable invariant manifold is found as a limit of successive approximations. Thus, the function ψ in (21) is a limit of the sequence of functions ψ_n defined inductively by the following rule

$$\psi_{n+1}(\bar{y}) = B\psi_n(y)[B + f(0, y)]^{-1} - f(0, y)[B + f(0, y)]^{-1}, \quad \psi_0 \equiv 0, \quad (25)$$

where $\bar{y}$ and y are related by (22). Hence, if we denote the matrix (20) as $u(x, y) \equiv -(I + \psi(y))^{-1}\psi'(y)x$, then $u(x, y)$ will be the limit of the sequence $u_n(x, y)$:

$$u_{n+1}(\bar{x}, \bar{y}) = \left(\frac{\partial \bar{x}}{\partial x} u_n(x, y) + \frac{\partial \bar{x}}{\partial y} \right) \left(\frac{\partial \bar{y}}{\partial y} \right)^{-1}, \quad u_0 \equiv 0 \quad (26)$$

where

$$\bar{x} = (B + f(0, y))x, \quad \bar{y} = (B^{-\top} + g(0, y))y. \quad (27)$$

⁶More precisely, it is a (2×2) -matrix whose entries are scalar functions of y .

Note that the map (27) is symplectic, the validity of (6) follows from the symplecticity of the original map (12), see (13), (14). Therefore, by Property 3.3, if u_n is a symmetric matrix, then u_{n+1} is symmetric too. Hence, the limit matrix (20) is symmetric and the formulas (18), (19) define a C^{r-1} -smooth symplectic coordinate transformation indeed.

Now note that the functional equation on ψ is designed in such a way that in the new coordinates (18) the map T_0 remains in the form (12) with some new functions f and g , and

$$\left. \frac{\partial \bar{x}}{\partial x} \right|_{x=0} \equiv B, \quad (28)$$

i. e. the new function f satisfies $f(0, y) \equiv 0$. It follows immediately from (13) that $g(0, y) = 0$, i. e. in the new coordinates the first two identities of (16) are fulfilled.

Absolutely analogously, one can construct a C^{r-1} -smooth symplectic coordinate transformation

$$x_{new} = \Phi(x), \quad y_{new} = (I + \phi(x))y \quad \text{with} \quad \Phi(0) = 0, \quad \Phi'(0) = I, \quad \phi(0) = 0, \quad (29)$$

such that the other two identities of (16) become valid (the first two identities cannot be destroyed by such transformation). ■

If the map under consideration depends on some parameters, continuously or smoothly, then the coordinate transformation we obtained in Lemma 1 depends on the parameters, respectively, continuously or smoothly too. It must however be noted that in the case of smooth parameter dependence the one last $(r-1)$ -th derivative with respect to the parameter may not exist for the functions ψ and ϕ defining the transformation. This follows just from construction, because the strong unstable and strong stable manifolds which we used in deriving our transformation depend on the parameters in the same manner; see [9] for more detail.

Moreover, we never used in the proof that the eigenvalues of B are complex. Obviously, the lemma remains valid in case B is a matrix, of an arbitrary dimension, *whose all eigenvalues have the same absolute value* — this condition is necessary and sufficient for the operator $z \mapsto BzB^{-1}$ to have all eigenvalues on the unit circle, which is crucial for the existence of the required strong unstable or strong stable manifolds.

The coordinates of Lemma 1 are quite convenient because if the identities (16) are fulfilled, all the iterations of the local map T_0 are uniformly close to the iterations of the linearized map. Namely, denote $(x_k, y_k) = T_0^k(x_0, y_0)$. It is well-known [10, 11, 12] that for sufficiently small δ , given any $k \geq 0$ and “the boundary data” x_0, y_k such that $\|x_0\| \leq \delta, \|y_k\| \leq \delta$, the corresponding orbit $(x_j, y_j)_{j=0}^k$ of T_0 is defined uniquely and lies in the small neighborhood U_0 of the fixed point $O(0, 0)$ entirely.

Applying Lemma 3.6 of [9] to the map under consideration gives the following result:

Lemma 2. *When identities (16) hold, the following relations are fulfilled:*

$$\begin{aligned} x_k &= B^k x_0 + \lambda^k p_k(x_0, y_k), \\ y_0 &= (B^\top)^k y_k + \lambda^k q_k(x_0, y_k) \end{aligned} \quad (30)$$

where p_k and q_k are C^{r-1} -functions which tend to zero along with all derivatives as $k \rightarrow +\infty$; here $\lambda \in (0, 1)$ is the absolute value of the eigenvalues of B , see (15).

Analogously to [13], these formulas may be enhanced as follows.

Lemma 3. *If $r \geq 3$ in Lemma 2, then*

$$\begin{aligned} x_k &= B^k x_0 + k\lambda^{2k} P_k(x_0, y_k)x_0, \\ y_0 &= (B^\top)^k y_k + k\lambda^{2k} Q_k(x_0, y_k)y_k \end{aligned} \quad (31)$$

where P_k and Q_k are uniformly bounded along with all derivatives up to the order $(r - 2)$.

The proof of this lemma repeats closely the proof of an analogous statement (Lemma 1.2 in [13]). Therefore, we prove here the boundedness only for the functions P_k and Q_k themselves; the boundedness of derivatives is verified along the same lines (for more detail see [13]).

We will use the method of the boundary-value problem [10, 11]. Consider the following operator $\mathcal{T}$:

$$\begin{aligned}\widehat{x}_j &= B^j x_0 + \sum_{s=0}^{j-1} B^{j-s-1} F(x_s, y_s), \\ \widehat{y}_j &= C^{j-k} y_k - \sum_{s=j}^{k-1} C^{j-s-1} G(x_s, y_s), \\ j &= 0, 1, \dots, k\end{aligned}\tag{32}$$

where we denote $C = B^{-\top}$; here, F and G are the functions from (8) which are estimated as

$$\|F\| \leq K \|x\|^2 \|y\|, \quad \|G\| \leq K \|x\| \|y\|^2,\tag{33}$$

according to (17). The operator $\mathcal{T}: [(x_j, y_j)]_{j=0}^k \mapsto [(\widehat{x}_j, \widehat{y}_j)]_{j=0}^k$ is defined on the set of sequences

$$R(\delta) = \{z = [(x_j, y_j)]_{j=0}^k, \|x_j\| \leq \delta, \|y_j\| \leq \delta\}$$

with the norm $\|z\| = \max_j \|(x_j, y_j)\|$. Note that if $[(x_j, y_j)]_{j=0}^k$ is a fixed point of $\mathcal{T}$, then

$$(x_0, y_0) \xrightarrow{T_0} (x_1, y_1) \xrightarrow{T_0} \dots \xrightarrow{T_0} (x_k, y_k),$$

i. e. the fixed point of $\mathcal{T}$ is the orbit of the map T_0 .

For sufficiently small δ_0 the operator $\mathcal{T}$ maps the set $R(\delta_0)$ into itself and is contracting on this set (see a proof in [12]). Thus, the map (32) has a unique fixed point which is the limit of iterations by $\mathcal{T}$ of any initial sequence in $R(\delta_0)$. Therefore, to get some estimates on the orbit of the map T_0 it is sufficient to show that the set of sequences $[(x_j, y_j)]_{j=0}^k$ satisfying these estimates lies within $R(\delta_0)$ and is invariant with respect to $\mathcal{T}$.

As such an invariant set, let us take $R_{x_0 y_k} \subset R(\delta_0)$ for which x_j, y_j satisfy the following inequalities (given x_0 and y_k):

$$\|x_j - B^j x_0\| \leq \delta_0 j \lambda^{j+k}, \quad \|y_j - C^{j-k} y_k\| \leq \delta_0 (k-j) \lambda^{2k-j}.\tag{34}$$

Note that these inequalities imply that for some constant K_0 , independent of δ_0 ,

$$\|x_j\| \leq K_0 \delta_0 \lambda^k, \quad \|y_j\| \leq K_0 \delta_0 \lambda^{k-j}.\tag{35}$$

Plugging (35), along with (33), in (32) we see that for δ_0 sufficiently small, if $z \in R_{x_0 y_k}$, then

$$\begin{aligned}\|\widehat{x}_j - B^j x_0\| &\leq K \sum_{s=0}^{j-1} \|B^{j-s-1} x_s\|^2 \|y_s\| \leq K K_0^3 \delta_0^3 \sum_{s=0}^{j-1} \lambda^{j-s-1} \lambda^{2s} \lambda^{k-s} = \\ &= \left(\frac{K K_0^3}{\lambda} \delta_0^2 \right) \delta_0 j \lambda^{j+k} < \delta_0 j \lambda^{j+k}, \\ \|y_j - C^{j-k} y_k\| &\leq K \sum_{s=j}^{k-1} \|C^{-1} x_s\| \|y_s\|^2 \leq K K_0^3 \delta_0^3 \sum_{s=j}^{k-1} \lambda^{s+1-j} \lambda^s \lambda^{2(k-s)} = \\ &= (K K_0^3 \lambda \delta_0^2) \delta_0 (k-j) \lambda^{2k-j} < \delta_0 (k-j) \lambda^{2k-j}.\end{aligned}$$

This means that $\widehat{z} \in R_{x_0 y_k}$. Thus, $\mathcal{T}(R_{x_0 y_k}) \subset R_{x_0 y_k}$.

So, we have proved that given x_0 and y_k , the corresponding orbit $[(x_j, y_j)]_{j=0}^k$ of T_0 is an element of $R_{x_0 y_k}$ (as this orbit is the fixed point of $\mathcal{T}$). By construction (compare (34) with (31)), this means that the functions P_k and Q_k (31) are uniformly bounded indeed. We skip the proof of boundedness of their derivatives.

When the map T_0 depends smoothly on some parameters, the matrix B and the functions p_k, q_k and P_k, Q_k in Lemmas 2 and 3 are also parameter dependent. The derivatives of these functions with respect to the parameters are estimated as follows (see Lemma B.1 in [9] and Lemma 1.2 in [13]): *the derivatives including s differentiations with respect to the parameters ($s = 1, \dots, r-2$) are of order $o(k^{-s})$ for functions p_k, q_k and $O(k^{-s})$ for P_k, Q_k .*

4. Global map T_1 near the orbit of homoclinic tangency

Now assume that the system X under consideration (a symplectic map or a Hamiltonian flow on the fixed energy level) has a homoclinic orbit Γ , at the points of which the stable and unstable manifolds of the saddle-focus periodic orbit are tangent. Moreover, we consider the case of the simplest tangency, i. e. such that satisfies conditions B.1, B.2 (see Section 1). Take a pair of points of Γ in U_0 : a point $M^+(x^+, 0) \in W_{loc}^s(O)$ and a point $M^-(0, y^-) \in W_{loc}^u(O)$ ⁷, where $x^+ = (x_1^+, x_2^+)$, $y^- = (y_1^-, y_2^-)$; obviously $(x_1^+)^2 + (x_2^+)^2 \neq 0$, $(y_1^-)^2 + (y_2^-)^2 \neq 0$.

All the forward iterations of M^+ by T_0 stay in U_0 on W_{loc}^s and tend to O ; all the backward iterations of M^- by T_0 stay in U_0 on W_{loc}^u and tend to O as well. Moreover, by construction, $X^t(M^-) = M^+$ for some positive t (where X^t denotes the time t shift by the orbits of X if X is a flow and it denotes the t -th power of X if it is a map).

Let Π^+ and Π^- be sufficiently small neighborhoods of, respectively, M^+ and M^- in U_0 . The map $T_1 : \Pi^- \rightarrow \Pi^+$ by the orbits of X close to the segment of Γ between M^- and M^+ is called *the global map*; when X is a map we just have $T_1 \equiv X^t|_{\Pi^-}$. If we denote the coordinates in Π^+ and Π^- as $(x^0, y^0) = (x_{01}, x_{02}, y_{01}, y_{02})$ and $(x^1, y^1) = (x_{11}, x_{12}, y_{11}, y_{12})$ respectively, then the global map is written as

$$\bar{x}^0 - x^+ = ax^1 + b(y^1 - y^-) + \dots, \quad \bar{y}^0 = cx^1 + d(y^1 - y^-) + \dots, \quad (36)$$

where the dots stand for the second and higher order terms; a, b, c and d are some (2×2) -matrices. Altogether, they comprise a symplectic (4×4) -matrix

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

hence they satisfy Property 3.2 from Section 2.

Note that the rotation in U_0

$$x_{new} = \mathcal{R}_\alpha x, \quad y_{new} = \mathcal{R}_\alpha y \quad (37)$$

with

$$\mathcal{R}_\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \quad (38)$$

is a symplectic transformation of coordinates. Moreover, it does not change the matrix B , neither it destroys identities (16), nor affects formulas (30), (31). Therefore, we are free to make the local rotation (37) with an arbitrary angle α and we will choose it so that to nullify as many entries of S as possible. The particular effect of such rotation on the block d of S is

$$d_{new} = \mathcal{R}_{-\alpha} d \mathcal{R}_\alpha. \quad (39)$$

⁷Recall that we have straightened the local invariant manifolds, so the equations of $W_{loc}^s(O)$ and $W_{loc}^u(O)$ in U_0 are $y_1 = y_2 = 0$ and $x_1 = x_2 = 0$ respectively.

By assumption, $T_1(W_{loc}^u)$ is tangent to W_{loc}^s at $M^+(x^+, 0)$ along a single vector. Thus, if we consider the linearization of the map T_1 at M^- :

$$\bar{x}^0 - x^+ = ax^1 + b(y^1 - y^-), \quad \bar{y}^0 = cx^1 + d(y^1 - y^-), \quad (40)$$

the image of the plane $\{x^1 = 0\}$ by this linearized map intersects the plane $\{y^0 = 0\}$ along a straight line. In other words, the equation

$$0 = d(y^1 - y^-) \quad (41)$$

has a one-parameter family of solutions. Thus,

$$\det d = 0 \quad \text{and} \quad \text{rank } d = 1. \quad (42)$$

This means that the rows of the matrix

$$d = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

are linearly dependent, but not all entries are zero. Now one can choose α in (39) such that the matrix d takes the form

$$d = \begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$$

where $d_{21}^2 + d_{22}^2 \neq 0$. Let us assume, for definiteness, that $d_{22} \neq 0$.

If this is not the case (i. e. if $d_{22} = 0$ but $d_{21} \neq 0$), then one may take a new pair of homoclinic points $(T_0^{-1}M^-, M^+)$ instead of (M^-, M^+) . The new map T_1 will be $T'_1 = T_1 T_0$ and, taking into account that the function g in the formula (12) for T_0 is identically zero on W_{loc}^u (see (16)), one can see that the corresponding matrix d will be

$$d' = \lambda^{-1} \begin{pmatrix} 0 & 0 \\ d_{21} & 0 \end{pmatrix} \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} = \lambda^{-1} \begin{pmatrix} 0 & 0 \\ d_{21} \cos \varphi & -d_{21} \sin \varphi \end{pmatrix}.$$

Since $\sin \varphi \neq 0$ (by assumption $\varphi \neq 0, \pi$), it follows that now $d'_{22} = -d_{21} \sin \varphi \neq 0$.

Summarizing, we may assume that the Jacobi matrix S for the global map T_1 at M^- has the following form:

$$S = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & 0 & 0 \\ c_{21} & c_{22} & d_{21} & d_{22} \end{pmatrix} \quad (43)$$

where $d_{22} \neq 0$. Since S is a symplectic matrix, it must satisfy relations (6) and (7), e. g.

$$\begin{aligned} a) & \quad b_{21}d_{22} - b_{22}d_{21} = 0, \\ b) & \quad c_{11}(b_{12}d_{21} - b_{11}d_{22}) = d_{22}, \\ c) & \quad c_{12}(b_{12}d_{21} - b_{11}d_{22}) = -d_{21}, \\ d) & \quad a_{21}d_{21} - b_{21}c_{21} = 1 + b_{11}c_{11}. \end{aligned} \quad (44)$$

Since $d_{22} \neq 0$, we have from (44.b)

$$c_{11} \neq 0, \quad b_{11}d_{22} - b_{12}d_{21} \neq 0. \quad (45)$$

Taking into account quadratic terms in the equation for $\bar{y}_{01}$ we can now write the map T_1 in the following form

$$\begin{aligned}\bar{x}_{01} - x_1^+ &= a_{11}x_{11} + a_{12}x_{12} + b_{11}(y_{11} - y_1^-) + b_{12}(y_{12} - y_2^-) + \dots, \\ \bar{x}_{02} - x_2^+ &= a_{21}x_{11} + a_{22}x_{12} + b_{21}(y_{11} - y_1^-) + b_{22}(y_{12} - y_2^-) + \dots, \\ \bar{y}_{01} &= c_{11}x_{11} + c_{12}x_{12} + D_1(y_{11} - y_1^-)^2 + D_2(y_{11} - y_1^-)(y_{12} - y_2^-) + D_3(y_{12} - y_2^-)^2 + \dots, \\ \bar{y}_{02} &= c_{21}x_{11} + c_{22}x_{12} + d_{21}(y_{11} - y_1^-) + d_{22}(y_{12} - y_2^-) + \dots.\end{aligned}\tag{46}$$

Since $x_{11} = x_{12} = 0$ on W_{loc}^u , its image $T_1 W_{loc}^u$ is given as follows (we used relations (44), (45)):

$$\begin{aligned}x_{02} - x_2^+ &= \frac{b_{22}}{d_{22}}y_{02} + \dots, \\ y_{01} &= \tilde{D}_1(x_{01} - x_1^+)^2 + \tilde{D}_2(x_{01} - x_1^+)y_{02} + \tilde{D}_3y_{02}^2 + \dots\end{aligned}\tag{47}$$

for some coefficients $\tilde{D}_j$. Condition B.2 requires the quadratic tangency of this surface to W_{loc}^s : $y_{01} = y_{02} = 0$, which is obviously equivalent to the non-vanishing of the coefficient $\tilde{D}_1$. One can compute that

$$\tilde{D}_1 = \frac{D_1 - D_2 \left(\frac{d_{21}}{d_{22}} \right) + D_3 \left(\frac{d_{21}}{d_{22}} \right)^2}{(b_{11} - b_{12} \frac{d_{21}}{d_{22}})^2}.$$

Thus, condition B.2 reads as

$$D_0 \equiv D_1 d_{22}^2 - D_2 d_{21} d_{22} + D_3 d_{21}^2 \neq 0.\tag{48}$$

We can always assume that $D_0 > 0$ (the sign of D_0 may be changed by changing the sign of all variables: $(x, y) \rightarrow (-x, -y)$).

Let us now include our system into a smooth one-parameter family X_μ . After transition to the coordinates of Lemma 1 the system is C^{r-1} -smooth with respect to phase variables (x, y) and C^{r-2} -smooth with respect to μ (more precisely, the first derivative with respect to (x, y) is C^{r-2} with respect to μ).

At non-zero values of the parameter μ the global map T_1 is written as

$$\begin{aligned}\bar{x}_{01} - x_1^+(\mu) &= a_{11}(\mu)x_{11} + a_{12}(\mu)x_{12} + b_{11}(\mu)(y_{11} - y_1^-) + b_{12}(\mu)(y_{12} - y_2^-) + \dots, \\ \bar{x}_{02} - x_2^+(\mu) &= a_{21}(\mu)x_{11} + a_{22}(\mu)x_{12} + b_{21}(\mu)(y_{11} - y_1^-) + b_{22}(\mu)(y_{12} - y_2^-) + \dots, \\ \bar{y}_{01} &= y_1^*(\mu) + c_{11}(\mu)x_{11} + c_{12}(\mu)x_{12} + d_{11}(\mu)(y_{11} - y_1^-) + d_{12}(\mu)(y_{12} - y_2^-) \\ &\quad + D_1(y_{11} - y_1^-)^2 + D_2(y_{11} - y_1^-)(y_{12} - y_2^-) + D_3(y_{12} - y_2^-)^2 + \dots, \\ \bar{y}_{02} &= y_2^*(\mu) + c_{21}(\mu)x_{11} + c_{22}(\mu)x_{12} + d_{21}(\mu)(y_{11} - y_1^-) + d_{22}(\mu)(y_{12} - y_2^-) + \dots,\end{aligned}\tag{49}$$

where the constant terms and coefficients of the linearized map are now C^{r-2} -functions of μ , the same concerns higher order terms denoted by ellipsis. Recall that at $\mu = 0$ we have that identities (44) are satisfied and

$$y_1^*(0) = y_2^*(0) = 0 \quad \text{and} \quad d_{11}(0) = d_{12}(0) = 0.$$

We assume that the dependence on μ is generic, in the sense that for μ from one side of zero, say, for $\mu > 0$, the homoclinic intersection of $T_1 W_{loc}^u$ with W_{loc}^s disappears and, moreover, the distance between these manifolds changes with non-zero velocity as μ varies. One can see that this is equivalent to requirement

$$\left. \frac{\partial y_1^*}{\partial \mu} \right|_{\mu=0} > 0.\tag{50}$$

Without loss of generality we will put

$$y_1^*(\mu) \equiv \mu \quad (51)$$

in (49).

Let us now enlarge X_μ to a two-parameter family $X_{\mu\varphi}$ for which the argument φ of the multipliers of the saddle-focus periodic orbit under consideration is taken as an independent parameter. In this case all functions of μ in (49) (except for $y_1^*(\mu) \equiv \mu$) are also C^{r-2} -functions of φ .

5. Reduction of the first return map to Hénon-like form

A single-circuit periodic orbit of $X_{\mu\varphi}$ corresponds to a fixed point of the *first return map* $T_i \equiv T_1 T_0^i$ for some i sufficiently large. The study of the first return maps is conducted here in the same way as it is done for the case of general systems with homoclinic tangencies [14, 5, 15]. The domain σ_i^0 of the map T_i consists of those points in a small neighborhood Π^+ of the homoclinic point $M^+(x^+, 0)$ whose i -th iteration by the local map T_0 lies in a small neighborhood Π^- of the homoclinic point $M^-(0, y^-)$. The domain is non-empty for large i : Lemma 2 describes σ_i^0 as a thin strip composed of points (x^0, y^0) such that $\|x^0 - x^+\| \leq \varepsilon$ for a sufficiently small ε and

$$y^0 = (B^\top)^i y^1 + \lambda^i q_i(x^0, y^1) \quad (52)$$

where y^1 ranges in $\|y^1 - y^-\| \leq \varepsilon$. It is seen from (52) and (15) that the strips σ_i^0 wind around the two-dimensional area $\Pi^+ \cap W_{loc}^s$, approaching it as $i \rightarrow +\infty$ (Fig. 1). Again by Lemma 2, the images $T_0^i \sigma_i^0$ in Π^- are the strips σ_i^1 winding onto $\Pi^- \cap W_{loc}^u$ as $i \rightarrow +\infty$. In greater detail, the general structure of the set of strips $\sigma_i^{0,1}$ is described in [15] or [9].

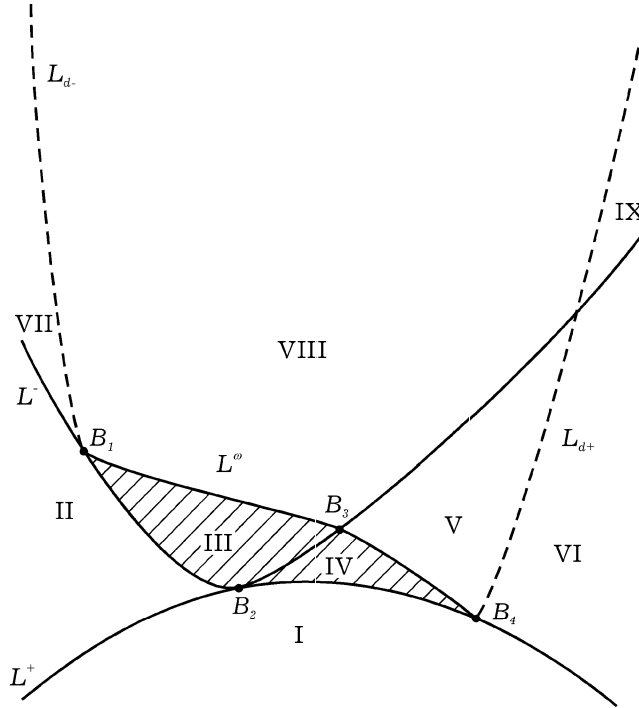


Fig. 1

Given i fixed, it is convenient to use (x^0, y^1) as the coordinates on the strip σ_i^0 where y^1 is related to (x^0, y^0) by (52). This change of coordinates is not symplectic. The symplectic form $dx^0 \wedge dy^0$ which is preserved by the map T_i , is written in the new coordinates as

$$dx^0 \wedge (((B^\top)^i + \lambda^i q'_{iy}(x^0, y^1)) dy^1 + \lambda^i q'_{ix}(x^0, y^1) dx^0),$$

see (52). Since the form is defined up to a scalar factor, we can divide to λ^i in this expression. Now, by (15) and since $q_i \rightarrow 0$ as $i \rightarrow +\infty$, we find that the map T_i in the new coordinates preserves the form

$$dx^0 \wedge \mathcal{R}_{-i\varphi} dy^1 + \dots \quad (53)$$

where the dots stand for terms vanishing as $i \rightarrow +\infty$, and $\mathcal{R}_\alpha$ denotes the rotation to the angle α (see (38)).

Lemma 4. *There exist infinitely many strips σ_i^0 for each of which the map T_i may be brought, by an affine transformation of coordinates and parameters $(\mu, \varphi) \mapsto (M_1, M_2)$, to the map asymptotically close (as $i \rightarrow +\infty$) to the four-dimensional Hénon-like map*

$$\begin{aligned} \bar{x}_2 &= x_1, & \bar{x}_1 &= y_2, & \bar{y}_2 &= y_1, \\ \bar{y}_1 &= M_1(y_1 + x_1) - x_2 - y_2^2 + M_2. \end{aligned} \quad (54)$$

Moreover, for any region $D \subset \mathbb{R}^2$ containing $(0, 0)$ there is $i_0 > 0$ such that the range of (M_1, M_2) covers D for $i > i_0$. The same is true for the range of new coordinates (x_1, x_2, y_1, y_2) .

Proof.

By (30), (15), (49) and (52) the map $T_i: (x^0, y^1) \mapsto (\bar{x}^0, \bar{y}^1)$ is written as

$$\begin{aligned} \begin{pmatrix} \bar{x}_{01} - x_1^+ \\ \bar{x}_{02} - x_2^+ \end{pmatrix} &= \lambda^i \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{01} \cos i\varphi - x_{02} \sin i\varphi \\ x_{01} \sin i\varphi + x_{02} \cos i\varphi \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} y_{11} - y_1^- \\ y_{12} - y_2^- \end{pmatrix} + \dots, \\ \lambda^i \begin{pmatrix} \bar{y}_{11} \cos i\varphi + \bar{y}_{12} \sin i\varphi \\ -\bar{y}_{11} \sin i\varphi + \bar{y}_{12} \cos i\varphi \end{pmatrix} &= \begin{pmatrix} \mu \\ y_2^* \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} x_{01} \cos i\varphi - x_{02} \sin i\varphi \\ x_{01} \sin i\varphi + x_{02} \cos i\varphi \end{pmatrix} + \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \cdot \\ &\cdot \begin{pmatrix} y_{11} - y_1^- \\ y_{12} - y_2^- \end{pmatrix} + \begin{pmatrix} D_1(y_{11} - y_1^-)^2 + D_2(y_{11} - y_1^-)(y_{12} - y_2^-) + D_3(y_{12} - y_2^-)^2 \\ \dots \end{pmatrix} + \dots, \end{aligned} \quad (55)$$

where the dots stand for $o(\lambda^i)$ terms (the terms $\lambda^i p_i$ and $\lambda^i q_i$ from (30); recall that $\lambda < 1$, so λ^i is small as $i \rightarrow +\infty$) and for the quadratic and higher order terms (other than those written down); the terms x_j^+ and y_2^* , the coefficients a_{jk} , b_{jk} , c_{jk} , d_{jk} and the terms denoted by the dots depend C^{r-2} -smoothly on μ and φ . Recall that the coefficients d_{11} and d_{12} are of order μ , as well as y_2^* .

We will shift the origin and also rotate the coordinate frame in order to make the free terms in the equation for $\bar{x}$ and in the second equation for $\bar{y}$ zero; moreover, the coefficients d_{11} and d_{12} must also vanish after this coordinate transformation. Precisely, we introduce new coordinates (ξ, η) :

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \mathcal{R}_{\alpha^*} \begin{pmatrix} x_{01} - x_1^+ - \xi_1^* \\ x_{02} - x_2^+ - \xi_2^* \end{pmatrix}, \quad \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{R}_{\alpha^*} \begin{pmatrix} y_{11} - y_1^- - \eta_1^* \\ y_{12} - y_2^- - \eta_2^* \end{pmatrix} \quad (56)$$

for some small (of order $O(|\mu| + \lambda^i)$) quantities ξ_1^* , ξ_2^* , η_1^* , η_2^* and α^* such that the map (55) recasts as

$$\begin{aligned} \begin{pmatrix} \bar{\xi}_1 \\ \bar{\xi}_2 \end{pmatrix} &= \lambda^i \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \cos i\varphi - \xi_2 \sin i\varphi \\ \xi_1 \sin i\varphi + \xi_2 \cos i\varphi \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \dots, \\ \lambda^i \begin{pmatrix} \bar{\eta}_1 \cos i\varphi + \bar{\eta}_2 \sin i\varphi \\ -\bar{\eta}_1 \sin i\varphi + \bar{\eta}_2 \cos i\varphi \end{pmatrix} &= \begin{pmatrix} \tilde{\mu} \\ 0 \end{pmatrix} + \lambda^i \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \cos i\varphi - \xi_2 \sin i\varphi \\ \xi_1 \sin i\varphi + \xi_2 \cos i\varphi \end{pmatrix} + \\ &+ \begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} D_1\eta_1^2 + D_2\eta_1\eta_2 + D_3\eta_2^2 \\ \dots \end{pmatrix} + \dots \end{aligned} \quad (57)$$

where the right-hand sides of the equation for $\bar{\xi}$ and of the second equation for $\bar{\eta}$ vanishes at $(\xi = 0, \eta = 0)$; the free term in the first equation for $\bar{\eta}$ is

$$\begin{aligned} \tilde{\mu} &= \mu + \lambda^i \{x_1^+ (c_{11} \cos i\varphi + c_{12} \sin i\varphi) + x_2^+ (c_{12} \cos i\varphi - c_{11} \sin i\varphi) - \\ &- y_1^- \cos i\varphi - y_2^- \sin i\varphi\} + o(|\mu| + \lambda^i). \end{aligned} \quad (58)$$

The dots in (57) stand for linear terms of order $o(\lambda^i)^8$ and for quadratic and higher order terms. The conditions of vanishing of the three free terms and two coefficients of the matrix d constitute a system of equations on ξ_1^* , ξ_2^* , η_1^* , η_2^* and α^* . The solvability of this system at small μ and λ^i is guaranteed by condition $D_0 \neq 0$ (see (48)⁹); we leave details to the reader.

The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ depends on μ , φ and, now, on i . By construction, this matrix is obtained by the linearization of the global map T_1 at the point

$$(x^1 = B^i(x^+ + \xi^*) + \lambda^i p_i(x^+ + \xi^*, y^- + \eta^*), y^1 = y^- + \eta^*)$$

plus the rotation of the type (37) on the angle α^* . Thus, this matrix is symplectic and the identities (44) are satisfied now for non-zero μ and any large i .

Note also that the transition to the coordinates (ξ, η) by formula (56) does not change the structure of the symplectic form (53). Thus, in the new coordinates the map T_i preserves the form

$$d\xi \wedge \mathcal{R}_{-i\varphi} d\eta + o(1)_{i \rightarrow +\infty}. \quad (59)$$

Now make the following linear transformation of coordinates:

$$\begin{aligned} u_1 &= \xi_1, & u_2 &= \xi_2, \\ v_1 &= d_{21}\eta_1 + d_{22}\eta_2 + \lambda^i \tilde{c}_{21}\xi_1, & v_2 &= -\eta_1 \sin i\varphi + \eta_2 \cos i\varphi \end{aligned} \quad (60)$$

where $\tilde{c}_{jk}$ denotes the corresponding entry of the matrix $\tilde{c} = c \cdot \mathcal{R}_{i\varphi}$. We will also use further the notation $\tilde{a} = a \cdot \mathcal{R}_{i\varphi}$, $\tilde{b} = b \cdot \mathcal{R}_{i\varphi}$, $\tilde{d} = d \cdot \mathcal{R}_{i\varphi}$. Note that the matrix $\tilde{S} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$ is symplectic (it satisfies (6)); moreover, it has the same structure as the matrix S in (43) — the first row of $\tilde{d}$ is zero. Therefore, $\tilde{S}$ must satisfy identities of kind (44); in particular,

$$\tilde{a}_{21}\tilde{d}_{21} - \tilde{b}_{21}\tilde{c}_{21} = 1 + \tilde{b}_{11}\tilde{c}_{11}. \quad (61)$$

The Jacobian of the transformation (60) equals to

$$\tilde{d}_{21} \equiv d_{21} \cos i\varphi + d_{22} \sin i\varphi.$$

Given i , this quantity is non-zero for all φ except for a number of special values. Fortunately, these exceptional values do not belong to the region we are interested in, as we will show below.

The inverse transformation to (60) is

$$\begin{aligned} \xi_1 &= u_1, & \xi_2 &= u_2, \\ \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \frac{1}{\tilde{d}_{21}} \begin{pmatrix} \cos i\varphi & -d_{22} \\ \sin i\varphi & d_{21} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \lambda^i \frac{\tilde{c}_{21}}{\tilde{d}_{21}} \begin{pmatrix} \cos i\varphi \\ \sin i\varphi \end{pmatrix} u_1. \end{aligned} \quad (62)$$

The symplectic form (59) is written in the new coordinates as

$$du_1 \wedge (dv_1 - \tilde{d}_{22}dv_2) + \tilde{d}_{22}du_2 \wedge dv_2 + o(1)_{i \rightarrow +\infty}. \quad (63)$$

⁸If the original map was at least C^4 , these terms are of order $O(i\lambda^{2i})$, see Lemma 3.

⁹This is the formal expression for the requirement that the tangency of $T_1 W_{\text{loc}}^u$ with W_{loc}^s at $\mu = 0$ is quadratic.

The map (57) recasts as follows (we use the first identity of (44)):

$$\begin{aligned}\bar{u}_1 &= \lambda^i \left(\tilde{a}_{11} - \frac{\tilde{b}_{11}\tilde{c}_{21}}{\tilde{d}_{21}} \right) u_1 + \tilde{a}_{12}u_2 + \frac{1}{\tilde{d}_{21}}(\tilde{b}_{11}v_1 + b_0v_2) + \dots, \\ \bar{u}_2 &= \lambda^i \left(\tilde{a}_{21} - \frac{\tilde{b}_{21}\tilde{c}_{21}}{\tilde{d}_{21}} \right) u_1 + \tilde{a}_{22}u_2 + \frac{\tilde{b}_{21}}{\tilde{d}_{21}}v_1 + \dots, \\ \frac{\lambda^i}{\tilde{d}_{21}}(\bar{v}_1 - \tilde{d}_{22}\bar{v}_2) &= \tilde{\mu} + \lambda^i(\tilde{c}_{11}u_1 + \tilde{c}_{12}u_2) + D_0v_2^2 + \dots, \\ \lambda^i\bar{v}_2 &= v_1 + \lambda^i\tilde{c}_{22}u_2 + \dots.\end{aligned}\tag{64}$$

Here $b_0 = b_{12}d_{21} - b_{11}d_{22}$, it is non-zero by virtue of (45); the non-zero quantity D_0 is defined by (48); the dots stand for linear terms of order $o(\lambda^i)$ (of order $O(i\lambda^{2i})$ if $r \geq 4$) and irrelevant quadratic and higher order terms.

Let us now rescale the coordinates:

$$\begin{aligned}v_1 &= -y_1\lambda^{3i}\frac{1}{D_0\tilde{d}_{21}}, & v_2 &= -y_2\lambda^{2i}\frac{1}{D_0\tilde{d}_{21}}, \\ u_1 &= -x_1\lambda^{2i}\frac{1}{D_0\tilde{d}_{21}}\frac{b_0}{\tilde{d}_{21}}, & u_2 &= -x_2\lambda^{3i}\frac{1}{D_0\tilde{d}_{21}}\frac{b_0}{\tilde{d}_{21}}.\end{aligned}\tag{65}$$

Using identities (44.b), (44.c), (61) one can see that (64) is now transformed to

$$\begin{aligned}\bar{x}_1 &= y_2 + \dots, \\ \bar{x}_2 &= (1 + \beta_1)x_1 + \beta_2y_1 + \dots, \\ \bar{y}_2 &= y_1 + \dots, \\ \bar{y}_1 &= M_1^{(1)}x_1 + M_1^{(2)}y_1 - x_2 + \widetilde{M}_2 - y_2^2\dots,\end{aligned}\tag{66}$$

whereas the symplectic form (63) is rewritten as

$$dx_1 \bigwedge (dy_1 - M_1^{(3)}dy_2) + dx_2 \bigwedge dy_2 + \dots.\tag{67}$$

In these formulas, the dots stand for the terms vanishing as $i \rightarrow +\infty$; the coefficients β_j are

$$\beta_1 = \frac{\tilde{b}_{11}}{b_0}\tilde{d}_{22}, \quad \beta_2 = \frac{\tilde{b}_{21}}{b_0}\tilde{d}_{21};\tag{68}$$

the quantities $M_1^{(1)}$, $M_1^{(2)}$, $M_1^{(3)}$ coincide in the main order¹⁰:

$$M_1^{(1,2,3)} = \lambda^{-i}(\tilde{d}_{22} + o(1)_{i \rightarrow +\infty});\tag{69}$$

finally,

$$\widetilde{M}_2 = -\lambda^{-4i}\tilde{d}_{21}^2D_0\tilde{\mu}.\tag{70}$$

We shall consider the map (66) for the values of φ and μ corresponding to bounded $\widetilde{M}_2$ and $M_1^{(3)}$. So, μ tends to zero as $i \rightarrow +\infty$. Moreover, $\tilde{d}_{22} \equiv d_{22} \cos i\varphi - d_{21} \sin i\varphi$ must tend to zero either. Note that while φ varies in some small interval near φ_0 , the value of $i\varphi$ runs a large interval at i sufficiently large. Therefore, for all large i there are values of φ close to φ_0 , corresponding to $\tilde{d}_{22}$ as

¹⁰The $o(1)$ components appear here as the result of rescaling of linear terms of order $o(\lambda^i)$ denoted by dots in (64); if the original map was at least C^4 , one may replace $o(1)$ by $O(i\lambda^i)$.

small as necessary. Note here that when $\tilde{d}_{22}$ is small, the value of $\tilde{d}_{21} \equiv d_{21} \cos i\varphi + d_{22} \sin i\varphi$ is of order $\sqrt{d_{21}^2 + d_{22}^2}$; i. e. the determinant of the coordinate transformation (60) is bounded away from zero indeed, as required.

Since the map (66) preserves the symplectic form (67), it follows immediately that when $M_1^{(3)}$ is bounded, the differences between the coefficients $M_1^{(1)}$, $M_1^{(2)}$ and $M_1^{(3)}$ must tend to zero as $i \rightarrow +\infty$. Thus, (66) is rewritten as

$$\begin{aligned}\bar{x}_1 &= y_2 + \dots, \\ \bar{x}_2 &= x_1 + \beta_2 y_1 + \dots, \\ \bar{y}_2 &= y_1 + \dots, \\ \bar{y}_1 &= M_1(x_1 + y_1) - x_2 + \tilde{M}_2 - y_2^2 + \dots\end{aligned}\tag{71}$$

where $M_1 = M_1^{(1)}$; we also drop the coefficient β_1 because it is proportional to $\tilde{d}_{22}$ (see (68)) and, hence, it tends to zero as $i \rightarrow +\infty$.

Finally, the transformation

$$x_2^{new} = x_2 - \beta_2 y_2 + \frac{\beta_2}{2}, \quad x_1^{new} = x_1 + \frac{\beta_2}{2}, \quad y_1^{new} = y_1 + \frac{\beta_2}{2}, \quad y_2^{new} = y_2 + \frac{\beta_2}{2}$$

brings the map to

$$\begin{aligned}\bar{x}_1 &= y_2 + \dots, \\ \bar{x}_2 &= x_1 + \dots, \\ \bar{y}_2 &= y_1 + \dots, \\ \bar{y}_1 &= M_1(x_1 + y_1) - x_2 + M_2 - y_2^2 + \dots\end{aligned}\tag{72}$$

where

$$M_2 = \tilde{M}_2 + \beta_2(1 - M_1) + \frac{\beta_2^2}{4}.$$

Note that one can make one more coordinate transformation, asymptotically close to identity as $i \rightarrow +\infty$ (it, therefore, does not change the form (72) of the map), such that the symplectic form (67) reduces to

$$dx_1 \wedge (dy_1 - M_1 dy_2) + dx_2 \wedge dy_2\tag{73}$$

for all sufficiently large i .

Summarizing, we have reduced the first return map T_i to the form (72), i. e. we have made it asymptotically close, along with $(r-1)$ derivatives with respect to (x, y) and $(r-2)$ derivatives with respect to M_1 and M_2 , to the Hénon-like map (54). The final formulas for M_1 and M_2 are¹¹

$$\begin{aligned}M_1 &= \lambda^{-i}(d_{22} \cos i\varphi - d_{21} \sin i\varphi + o(1)), \\ M_2 &= -\lambda^{-4i}(d_{21}^2 + d_{22}^2)D_0(\mu + \\ &\quad + \lambda^i \frac{1}{\sqrt{d_{21}^2 + d_{22}^2}} \{x_2^+(c_{12}d_{21} - c_{11}d_{22}) - y_1^- d_{21} - y_2^- d_{22}\} + o(\lambda^i)).\end{aligned}\tag{74}$$

Obviously, one can always find the regions arbitrarily close to $(\mu = 0, \varphi = \varphi_0)$ such that for respectively large i , the range of (M_1, M_2) in these regions includes all sufficiently large values. The lemma is proven. ■

¹¹We use (58), (69), (70) along with identities (44) and also take the smallness of $d_{22} \cos i\varphi - d_{21} \sin i\varphi$ into account.

6. Bifurcations of the fixed points in the rescaled map

The main (codimension one) local bifurcations of symplectic maps are [3]:

- 1) bifurcations of a fixed point with the double multiplier $(+1)$;
- 2) bifurcations of a fixed point with the double multiplier (-1) ;
- 3) bifurcations of a fixed point with a complex conjugate pair of double multipliers on the unit circle (i. e. with the multipliers of the kind $\nu_{1,2} = \nu_{3,4} = e^{\pm i\omega}$, $\omega \neq 0, \pi$).

In a sense, these bifurcations correspond, respectively, to a saddle-node bifurcation, period-doubling bifurcation and Andronov-Hopf bifurcation in general dynamical systems.

In the Hénon-like map

$$\begin{aligned}\bar{x}_2 &= x_1, & \bar{x}_1 &= y_2, & \bar{y}_2 &= y_1, \\ \bar{y}_1 &= -x_2 + M_1(x_1 + y_1) + M_2 - y_2^2\end{aligned}\tag{75}$$

all three bifurcations are encountered. We denote the corresponding bifurcation curves on the plane M_1 and M_2 as, respectively, L^+ , L^- and L^ω . To find these curves, note first that it follows from (75) that the coordinates of a fixed point satisfy $x_1 = x_2 = y_1 = y_2 = z$ where

$$z^2 + 2z(1 - M_1) - M_2 = 0.\tag{76}$$

The characteristic equation at the fixed point is

$$\nu^4 - M_1\nu^3 + 2z\nu^2 - M_1\nu + 1 = 0.\tag{77}$$

By (76)–(77), the curve $L^+ : \nu_1 = \nu_2 = 1$ is given by the following system

$$\begin{aligned}z^2 + 2z(1 - M_1) - M_2 &= 0, \\ 1 - M_1 + z &= 0,\end{aligned}\tag{78}$$

which recasts as

$$L^+ : M_2 = -(M_1 - 1)^2.\tag{79}$$

The curve $L^- : \nu_1 = \nu_2 = -1$ is given by

$$\begin{aligned}z^2 + 2z(1 - M_1) - M_2 &= 0, \\ 1 + M_1 + z &= 0,\end{aligned}\tag{80}$$

which transforms to

$$L^- : M_2 = (M_1 + 1)(3M_1 - 1).\tag{81}$$

For the curve $L^\omega : \nu_{1,2} = \nu_{3,4}^* = e^{i\omega}$ we have

$$\begin{aligned}z^2 + 2z(1 - M_1) - M_2 &= 0, \\ M_1 &= 4 \cos \omega, & z &= 2 + \cos 2\omega,\end{aligned}\tag{82}$$

which may be rewritten as

$$L^\omega : M_2 = \frac{1}{8}\left(1 + \frac{M_1^2}{8}\right)(M_1^2 - 16M_1 + 24) \quad \text{and} \quad |M_1| < 4.\tag{83}$$

Note that at $|M_1| > 4$ the same equation (83) defines some auxiliary (non-bifurcational) curves L_{d+} and L_{d-} which correspond to the fixed point with double real multipliers: $\nu_{1,2} = \gamma$, $\nu_{3,4} = \gamma^{-1}$. The

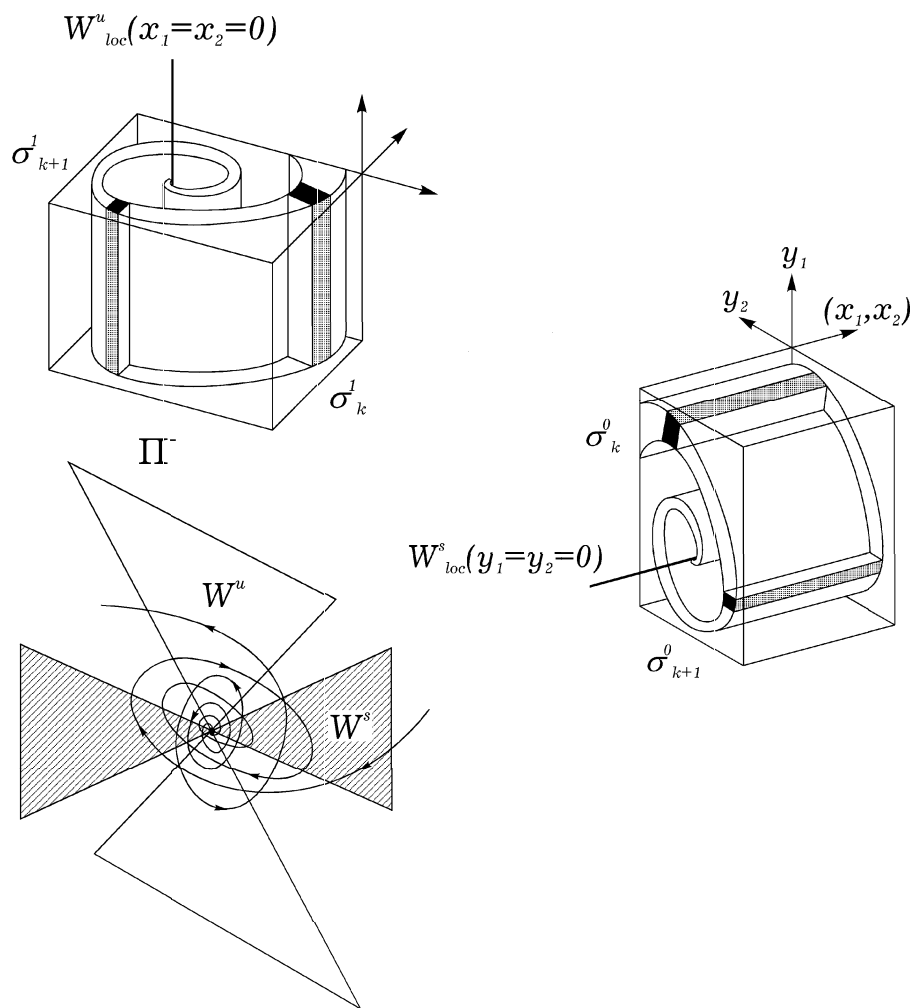


Fig. 2.

curve L_{d+} lies in the region $M_1 > 4$ and corresponds to positive γ , and the curve L_{d-} lies in $M_1 < -4$ and corresponds to negative γ .

The bifurcation diagram is shown in Fig. 2. We see that the plane (M_1, M_2) is divided into 9 regions by the curves L_+ , L_- , L_ω and $L_{d\pm}$.

At M_1, M_2 from *region I* the map (75) has no fixed points. In the other regions it has exactly two fixed points of the following types.

Region II. A saddle $(+, -)$ (i. e. a pair of positive and a pair of negative real multipliers) and a $(1-)$ elliptic point (i. e. a pair of multipliers on the unit circle and a pair of real negative multipliers).

Region III. A saddle $(+, -)$ and a 2-elliptic point. *Region IV.* A $(1+)$ elliptic point and a 2-elliptic point.

Region V. A $(1+)$ elliptic point and a saddle-focus.

Region VI. A saddle $(+, +)$ and a $(1+)$ elliptic point.

Region VII. A saddle $(+, -)$ and a saddle $(-, -)$.

Region VIII. A saddle $(+, -)$ and a saddle-focus.

Region IX. A saddle $(+, -)$ and a saddle $(-, -)$.

Note also 4 codimension-two points: B_1 corresponds to a fixed point with the multipliers $(-1, -1, -1, -1)$; B_2 corresponds to a fixed point with the multipliers $(-1, -1, +1, +1)$; B_4 corresponds to a fixed point with the multipliers $(+1, +1, +1, +1)$; and B_3 corresponds to one fixed point with the double multiplier (-1) and the other fixed point with a complex conjugate pair of

multipliers on the unit circle.

In the regions III and IV (the dashed curvilinear triangle without the segment (B_2, B_3) in Fig. 2) the map (75) has a 2-elliptic fixed point. Let us show that the 2-elliptic fixed point is generic at almost all points in this region, except for some finite number of curves.

Indeed, let $x_1 = x_2 = y_1 = y_2 = z$ be an elliptic fixed point of (75) with the multipliers $(e^{\pm i\omega_1}, e^{\pm i\omega_2})$. Let us denote $\xi = (x_2 - z, x_1 - z, y_2 - z, y_1 - z)$. Then the map (75) is written as

$$\bar{\xi} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & M_1 & 2z & M_1 \end{pmatrix} - \xi \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \xi_3^2. \quad (84)$$

The complex eigenvector corresponding to a multiplier $e^{i\omega}$ is, obviously, $(1, e^{i\omega}, e^{2i\omega}, e^{3i\omega})$. Therefore, if $\omega_1 \neq \omega_2$, the linear part is diagonalized after the transition to the (complex) coordinates (u, v) defined as follows:

$$\xi = u \begin{pmatrix} 1 \\ e^{i\omega_1} \\ e^{2i\omega_1} \\ e^{3i\omega_1} \end{pmatrix} + u^* \begin{pmatrix} 1 \\ e^{-i\omega_1} \\ e^{-2i\omega_1} \\ e^{-3i\omega_1} \end{pmatrix} + v \begin{pmatrix} 1 \\ e^{i\omega_2} \\ e^{2i\omega_2} \\ e^{3i\omega_2} \end{pmatrix} + v^* \begin{pmatrix} 1 \\ e^{-i\omega_2} \\ e^{-2i\omega_2} \\ e^{-3i\omega_2} \end{pmatrix}, \quad (85)$$

where the stars denote complex conjugation. In particular, the third component ξ_3 of ξ is

$$\xi_3 = ue^{2i\omega_1} + u^*e^{-2i\omega_1} + ve^{2i\omega_2} + v^*e^{-2i\omega_2}.$$

In the new coordinates the map is written as

$$\begin{aligned} \bar{u} &= e^{i\omega_1}u + \frac{i}{4\sin\omega_1(\cos\omega_1 - \cos\omega_2)}e^{-i\omega_1}(ue^{2i\omega_1} + u^*e^{-2i\omega_1} + ve^{2i\omega_2} + v^*e^{-2i\omega_2})^2, \\ \bar{v} &= e^{i\omega_2}v + \frac{i}{4\sin\omega_2(\cos\omega_2 - \cos\omega_1)}e^{-i\omega_2}(ue^{2i\omega_1} + u^*e^{-2i\omega_1} + ve^{2i\omega_2} + v^*e^{-2i\omega_2})^2. \end{aligned} \quad (86)$$

To reduce the map to the normal form we must eliminate all quadratic terms. It is done here by the standard normalizing transformation:

$$\begin{aligned} u_{new} &= u + \frac{i}{4\sin\omega_1(\cos\omega_1 - \cos\omega_2)}e^{-i\omega_1}W(u, v, \omega_1, \omega_2), \\ v_{new} &= v + \frac{i}{4\sin\omega_2(\cos\omega_2 - \cos\omega_1)}e^{-i\omega_2}W(v, u, \omega_2, \omega_1), \end{aligned} \quad (87)$$

where

$$\begin{aligned} W(u, v) &= u^2 \frac{e^{4i\omega_1}}{e^{i\omega_1} - e^{2i\omega_1}} + uu^* \frac{2}{e^{i\omega_1} - 1} + u^{*2} \frac{e^{-4i\omega_1}}{e^{i\omega_1} - e^{-2i\omega_1}} + v^2 \frac{e^{4i\omega_2}}{e^{i\omega_1} - e^{2i\omega_2}} + \\ &+ vv^* \frac{2}{e^{i\omega_1} - 1} + v^{*2} \frac{e^{-4i\omega_2}}{e^{i\omega_1} - e^{-2i\omega_2}} + 2uv \frac{e^{2i(\omega_1+\omega_2)}}{e^{i\omega_1} - e^{i(\omega_1+\omega_2)}} + \\ &+ 2u^*v^* \frac{e^{-2i(\omega_1+\omega_2)}}{e^{i\omega_1} - e^{-i(\omega_1+\omega_2)}} + 2uv^* \frac{e^{2i(\omega_1-\omega_2)}}{e^{i\omega_1} - e^{i(\omega_1-\omega_2)}} + 2u^*v \frac{e^{2i(\omega_2-\omega_1)}}{e^{i\omega_1} - e^{i(\omega_2-\omega_1)}}. \end{aligned} \quad (88)$$

Note that we assume additionally here that there is no strong resonances of the kind $\omega_{1,2} = 2\pi/3$, $\omega_1 = 2\omega_2$, $\omega_1 + 2\omega_2 = 2\pi$, $2\omega_1 + \omega_2 = 2\pi$, $2\omega_1 = \omega_2$.

After the transformation the map takes the form (we omit lengthy computations)

$$\begin{aligned}\bar{u} &= e^{i\omega_1} u \left(1 + \frac{i}{8 \sin \omega_1 (\cos \omega_1 - \cos \omega_2)^2} \{ u u^* \Omega_1(\omega_1, \omega_2) + v v^* \Omega_2(\omega_1, \omega_2) \} \right) + \dots, \\ \bar{v} &= e^{i\omega_2} v \left(1 + \frac{i}{8 \sin \omega_2 (\cos \omega_2 - \cos \omega_1)^2} \{ v v^* \Omega_1(\omega_2, \omega_1) + u u^* \Omega_2(\omega_2, \omega_1) \} \right) + \dots,\end{aligned}\tag{89}$$

where

$$\begin{aligned}\Omega_1(\omega_1, \omega_2) &= \frac{1}{\cos 2\omega_1 + \cos \omega_2} + \frac{2}{1 - \cos \omega_2} - \frac{1 + 4 \cos \omega_1}{(1 - \cos \omega_1)(1 + 2 \cos \omega_1)}, \\ \Omega_2(\omega_1, \omega_2) &= \frac{2}{1 - \cos \omega_2} - \frac{2}{1 - \cos \omega_1} + \frac{4 \cos \omega_2}{\cos \omega_1 + 2 \cos \omega_2} + \frac{4 \cos \omega_1}{\cos 2\omega_1 + \cos \omega_2}.\end{aligned}\tag{90}$$

The dots in (89) stand for other cubic and higher order terms. If we assume more non-resonance conditions: $\omega_{1,2} \neq \pi/2$, $\omega_1 \neq 3\omega_2$, $\omega_2 \neq 3\omega_1$, $\omega_1 + \omega_2 \neq \pi$, $3\omega_1 \pm \omega_2 \neq 2\pi$, $3\omega_2 \pm \omega_1 \neq 2\pi$, then all the cubic terms, not presented in formula (89), are non-resonant and can be killed by the further normalizing transformation which does not change the form (89) of the map. So, we may assume that there is no other cubic terms in (89). Thus, in symplectic polar coordinates

$$u = \sqrt{8\rho_1} e^{i\theta_1} |\cos \omega_2 - \cos \omega_1|, \quad v = \sqrt{8\rho_2} e^{i\theta_2} |\cos \omega_2 - \cos \omega_1|,$$

the map is written as

$$\bar{\rho} = \rho(1 + o(\rho)), \quad \bar{\theta} = \theta + \omega + \Omega\rho + o(\rho)\tag{91}$$

where $\rho = (\rho_1, \rho_2)$, $\theta = (\theta_1, \theta_2)$, $\omega = (\omega_1, \omega_2)$ and

$$\Omega = \frac{1}{\sin \omega_1 \sin \omega_2} \begin{pmatrix} \sin \omega_2 \Omega_1(\omega_1, \omega_2) & \sin \omega_2 \Omega_2(\omega_1, \omega_2) \\ \sin \omega_1 \Omega_2(\omega_2, \omega_1) & \sin \omega_1 \Omega_1(\omega_2, \omega_1) \end{pmatrix}.$$

The elliptic point is generic if $\det \Omega \neq 0$, i. e. if

$$\Omega_1(\omega_1, \omega_2) \Omega_1(\omega_2, \omega_1) - \Omega_2(\omega_1, \omega_2) \Omega_2(\omega_2, \omega_1) \neq 0.\tag{92}$$

The left-hand side is some rational function of $\cos \omega_1$ and $\cos \omega_2$ (see (90)), so it may vanish only on a number of curves in (ω_1, ω_2) -plane, if it is not identically zero. It is seen from (90) that if we fix some ω_2 , then at ω_1 close to zero the left-hand side of (92) is of order

$$\sim 7 \frac{1}{3} \frac{1}{(1 - \cos \omega_1)^2},$$

so it is non-zero indeed.

Thus, the 2-elliptic fixed point is generic for almost all (ω_1, ω_2) , hence, for almost all (M_1, M_2) from the region $III \cup IV$.

Let us now prove the main theorem (see Section 1). Note that a generic elliptic point of a symplectic map remains generic for any close symplectic map. Thus, by virtue of Lemma 4, for any sufficiently large i , the first return map T_i has a generic 2-elliptic fixed point in some region Δ_i in (μ, φ) -plane close to the point $(\mu = 0, \varphi = \varphi_0)$. This completes the proof.

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References

- [1] *S. V. Gonchenko, L. P. Shilnikov.* On two-dimensional analytic area-preserving diffeomorphisms with infinitely many stable elliptic periodic points. *Reg. and Chaot. Dyn.* 1997. V. 2. № 3/4. P. 106–123.
- [2] *S. V. Gonchenko, L. P. Shilnikov.* On homoclinic bifurcations for two-dimensional area-preserving diffeomorphisms. To appear.
- [3] *V. I. Arnold.* Mathematical methods of the classical mechanics. M.: Nauka. 1974.
- [4] *H. Russman.* Kleine Nenner I, Über invariante Kurven differenzierbarer Abbildungen eines Kreistrings. *Nachr. Akad. Wiss. Gott. Math. Phys. Kl.* 1970. V. II. P. 67–105.
- [5] *S. V. Gonchenko, L. P. Shilnikov, D. V. Turaev.* Dynamical phenomena in multidimensional systems with a structurally unstable homoclinic Poincaré curve. *Russian Acad. Sci. Dokl. Math.* 1993. V. 47. P. 410–415.
- [6] *D. V. Turaev.* On dimension of non-local bifurcational problems. *Bifurcation and Chaos.* 1996. V. 6.
- [7] *L. M. Lerman, Ya. L. Uman'skii.* On smooth canonical changes of variables. In *Methods of Qualitative Theory of Diff. Equations.* Gorki St. Univ. 1984. P. 140–147. (in Russian).
- [8] *P. Lancaster.* Theory of matrices. Academic Press. New-York – London. 1969.
- [9] *L. P. Shilnikov, A. L. Shilnikov, D. V. Turaev, L. O. Chua.* Methods of qualitative theory in nonlinear dynamics. Part I. World Scientific. Singapore. 1999.
- [10] *L. P. Shilnikov.* On a Poincaré–Birkhoff problem. *Math. USSR. Sbornik.* 1967. V. 3. P. 353–371.
- [11] *L. P. Shilnikov.* A contribution to the problem of structure of a neighbourhood of a homoclinic tube of invariant torus. *Soviet Math. Dokl.* 1968. V. 180. № 2. P. 286–289. (in Russian).
- [12] *V. S. Afraimovich, L. P. Shilnikov.* On critical sets of Morse–Smale systems. *Trudy Moskov. Mat. Obshch.* 1973. V. 28. P. 181–214; English transl. in *Trans. Moscow Math. Soc.* 1973. V. 28.
- [13] *S. V. Gonchenko, L. P. Shilnikov.* On moduli of systems with a structurally unstable homoclinic Poincaré curve. *Russian Acad. Sci. Izv. Math.* 1992. V. 41. P. 417–445.
- [14] *N. K. Gavrilov, L. P. Shilnikov.* On three-dimensional dynamical systems close to systems with a structurally unstable homoclinic curve. I. *Math. USSR Sbornik.* 1972. V. 17. P. 467–485; II. 1973. V. 19. P. 139–156.
- [15] *Gonchenko S. V., Shilnikov L. P., Turaev D. V.* Dynamical phenomena in systems with structurally unstable Poincaré homoclinic orbits. *Chaos.* 1996. V. 6. № 1. P. 15–31.

С. В. ГОНЧЕНКО, Д. В. ТУРАЕВ, Л. П. ШИЛЬНИКОВ

ЭЛЛИПТИЧЕСКИЕ ПЕРИОДИЧЕСКИЕ ОРБИТЫ ВБЛИЗИ ГОМОКЛИНИЧЕСКОГО КАСАНИЯ ДЛЯ ЧЕТЫРЕХМЕРНЫХ СИМПЛЕКТИЧЕСКИХ ОТОБРАЖЕНИЙ И ГАМИЛЬТОНОВЫХ СИСТЕМ С ТРЕМЯ СТЕПЕНЯМИ СВОБОДЫ

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Изучаются бифуркации, приводящие к возникновению эллиптических периодических траекторий в случае четырехмерных симплектических диффеоморфизмов (или гамильтоновых потоков с тремя степенями свободы), имеющих неглубокую гомоклиническую траекторию периодической орбиты типа седло-фокус.
